STABILITY OF THE UTILITY MAXIMIZATION PROBLEM WITH RANDOM ENDOWMENT IN INCOMPLETE MARKETS

CONSTANTINOS KARDARAS AND GORDAN ŽITKOVIĆ

Abstract. We perform a stability analysis for the utility maximization problem in a general semimartingale model where both liquid and illiquid assets (random endowments) are present. Small misspecifications of preferences (as modeled via expected utility), as well as views of the world or the market model (as modeled via subjective probabilities) are considered. Simple sufficient conditions are given for the problem to be well-posed, in the sense the optimal wealth and the marginal utility-based prices are continuous functionals of preferences and probabilistic views.

0. Introduction

0.1. **Expected Utility Maximization.** A mathematically sound, aesthetically pleasing and computationally tractable description of optimal behavior of rational economic agents under uncertainty comes from the *expected utility theory*: given a random outcome X (e.g., a terminal wealth, or a consumption stream) an agent's numerical assessment of the "satisfaction" that X provides is given by $\mathbb{E}[U(X)]$, where U is a real-valued function, and $\mathbb{E}$ is the expectation corresponding to either a *physically* estimated probability measure, or, in other circumstances, the *subjective* agent's view of the world.

Despite the criticism it received, expected utility theory has grown widely popular and successful, mainly because it delivers *quantitative* results and, in some cases, even closed-form solutions. Among the seminal contributions in this vein in the field of mathematical finance we single out [26] (dealing with a simple discrete-time Markovian model) and [23] (where the problem of optimal investment in a continuous-time Markovian framework is explicitly solved). A more general approach that avoids Markovian assumptions for the asset-price processes is the so-called martingale method. In complete financial markets, this methodology was introduced in [24] and later developed in [4], [5] and [15]. For incomplete financial models and continuous-time diffusion models important early progress was made in [9] and [16]. [19] and [20] contain a very complete picture of the solution of the problem of expected utility maximization from terminal wealth in a general semimartingale incomplete model when the wealth process remains positive.

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0.2. **Stability Analysis.** With problems of existence and uniqueness of optimal investment virtually settled (at least for utilities defined on the positive real line), interest in the *stability analysis* (for the solution of the problem of expected utility maximization under perturbations of various initial conditions) has recently developed. The problem of convergence of prices of illiquid assets, when the prices of the liquid assets converge, was tackled in [11]. In [14], the authors look at an Itô process model and a convergent sequence of utility functions (i.e., misspecifications of a "true" utility function). Convergence of utilities is also considered in [3], but in a general discrete-time setting. Continuity (and smoothness) properties with respect to perturbations in the initial wealth and the quantities of the illiquid assets have been studied in [18]. [21] deals with utility-function misspecifications in continuous-time models with general continuous-path semimartingale price processes, and illustrates the theory with applications to certain widely-used models. A different viewpoint is taken in [22]. Therein *model-*, rather than utility misspecifications are studied: the asset price process S^λ — a continuous-path semimartingale — is indexed by its *market-price-of-risk* λ (the parameter which is the source of model misspecification).

In view of the previously listed works, one can argue that there has been no unified treatment of the problem of stability under simultaneous perturbations of both the utility functions and the probability measures under which the expectations are taken. The aim of the present paper is to give insight into this problem in a general semimartingale model, where the economic agent is, additionally, endowed with a random payoff (illiquid assets). Moreover, rather than merely providing a common platform for most of the existing results, we generalize them in several directions. A simple sufficient (and, in some cases, "very close" to necessary) condition for stability is given, and several illuminating examples dealing with various special cases are provided. We remark that in this paper we deal with utility functions defined only on the positive real line, since the theory of utility maximization with random endowments for this case has been thoroughly understood. It would be interesting to pursue whether a treatment of stability for utility functions defined on the whole real line is possible, in the spirit of the recent developments of [2], but we are not dealing with this case in the present work. We also note that the results appearing here have qualitative nature and constitute a zeroth order approach to the problem. The next natural step would be a first-order study, quantifying the infinitesimal change of value functions, the optimal wealth, as well as utility indifference prices. This would be accomplished by a study of the differentiability of the latter outputs with respect to smooth changes of the preferences and the agent's subjective views. We leave this important task as a future research project.

The structure of the paper is simple. After this Introduction, section 1 describes the problem and states the main result, while all the proofs are given in section 2.

1. Problem Formulation and Statement of the Main Result

1.1. **Description of the modeling framework.** We start with a brief reproduction of the set-up and notation introduced in [13], where the authors are concerned with the problem of utility maximization with random endowment in incomplete semimartingale markets.

1.1.1. *The financial market.* Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be filtered probability space, where the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the *usual conditions* of right continuity and $\mathbb{P}$-completeness. The time horizon $T > 0$ is fixed and constant. This assumption is in place for simplicity only — T could be replaced by a finite stopping time, as is the case in [13] upon which we base our analysis.

We consider a financial market with d *liquid* assets, modeled by stochastic processes $S = (S^i)_{i=1,\dots,d}$. There is also a "baseline" asset S^0 which plays the role of a numéraire — this amounts to the standard assumption $S^0 \equiv 1$. The process S is assumed to be a locally bounded $\mathbb{R}^d$-valued semimartingale (see [6] for the economic justification of this essentially necessary assumption). Finally, in relation to the notion of absence of arbitrage, we posit the existence of at least one *equivalent martingale measure*, i.e., a probability measure $\mathbb{Q} \sim \mathbb{P}$ that makes (each component of) S a local martingale (see [6] and [8] for more information).

1.1.2. *Investment opportunities.* An initial capital $x > 0$ and a choice of an investment strategy H (assumed to be d-dimensional, predictable and S-integrable) result in a wealth process $X = X^{x,H} = x + H \cdot S$, where "$\cdot$" denotes *vector* stochastic integration. In order to avoid so-called doubling strategies, we restrict the class of investment strategies in a standard way: the wealth process $X = X^{x,H}$ is called *admissible* if

$$\mathbb{P}[X_t \geq 0,\ \forall\ 0 \leq t \leq T] = 1.$$

An admissible wealth process X is called *maximal* if for each $X' \in \mathcal{X}$ with $\mathbb{P}[X'_T \geq X_T] = 1$ and $X'_0 = X_0$, we necessarily have $X = X'$, a.s. The class of admissible wealth processes (starting from the initial wealth $X_0 = x$) is denoted by $\mathcal{X}(x)$. The union $\bigcup_{x>0} \mathcal{X}(x)$ is denoted by $\mathcal{X}$.

On the dual side, we define the class of *separating measures* by

$$\mathcal{Q} := \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P},\ \text{and } X \text{ is } \mathbb{Q}\text{-supermartingale for all } X \in \mathcal{X}\}.$$

Thanks to the assumptions of no-arbitrage and local boundedness, $\mathcal{Q}$ coincides with the set of all equivalent (local) martingale measures, and is, therefore, non-empty. For future use, we restate the (already imposed) assumption of *No Free Lunch with Vanishing Risk* as

$$\text{(NFLVR)} \qquad \mathcal{Q} \neq \emptyset.$$

1.1.3. *Illiquid assets.* Together with the liquid (traded) assets S, we assume the existence of N *illiquid* assets whose values at time T are represented by random variables $f^1, \dots, f^N$. We allow for the case $N = 0$, in which all assets are liquid. From the outset, the agents hold some positions in illiquid assets, but, due to their illiquidity, they are not able to trade in them (until the time T, at which all N of them mature). The only regularity assumption on the illiquid assets is that they

can be super- and sub- replicated using the traded assets S; in other words, we assume (with the convention that $\sum_{j=1}^{0} \cdot = 0$)

$$\text{(S-REP)} \qquad \mathcal{X}' := \Big\{ X \in \mathcal{X} \mid X_T \geq \sum_{j=1}^{N} |f^j|, \ \mathbb{P}\text{-a.s.} \Big\} \neq \emptyset.$$

To avoid trivial technical complications, we assume that the illiquid assets $f^1, \dots, f^N$ are non-redundant when $N \geq 1$, in the sense that no linear combination $\sum_{k=1}^{N} \alpha_k f^k$ — where not all of the α_k's are zero — is replicable in the sense that there exists a wealth process $X \in \mathcal{X}$ such that both X and $-X$ are maximal and $X_T = \sum_{k=1}^{N} \alpha_k f^k$. It is a standard result (see for example Lemma 7 in [13]) that this is equivalent to saying that the set of *arbitrage-free prices* for f defined as

$$\text{(N-TRAD)} \qquad \mathcal{P}(f) := \{ (\mathbb{E}^{\mathbb{Q}}[f^1], \dots, \mathbb{E}^{\mathbb{Q}}[f^N]) \,|\, \mathbb{Q} \in \mathcal{Q} \} \text{ is an open set when } N > 0.$$

If (N-TRAD) did not hold, we could always retain a minimal set of (linear combinations) of the illiquid claims, and regard all the others merely as outcomes of trading strategies using the liquid assets only. It should become clear that (N-TRAD) is not needed for the results of the paper to hold and this is why it is not assumed in our main Theorem 1.6 below.

Under the assumptions (NFLVR) and (S-REP), the class

$$\mathcal{Q}' := \{ \mathbb{Q} \in \mathcal{Q} \mid X \text{ is a } \mathbb{Q}\text{-uniformly integrable martingale for some } X \in \mathcal{X}' \}.$$

can be shown to be non-empty. This follows from the fact that $\mathcal{X}'$ contains at least one maximal element X. The existence of a measure $\mathbb{Q} \in \mathcal{Q}$ that makes this maximal wealth process a uniformly integrable martingale was established in [7]. Assumption (S-REP) implies that $f^j \in \mathbf{L}^1(\mathbb{Q})$ for all $j = 1, \dots, N$, $\mathbb{Q} \in \mathcal{Q}'$.

1.1.4. *Acceptability requirements.* The notion of acceptability, related to that of maximality introduced above, plays a natural role when non-bounded random endowment is present, as is thoroughly explained in [7] and [13]. We say that a process $X = X^{x,H} = x + H \cdot S$, with H predictable and S-integrable, is *acceptable*, if there exists a maximal wealth process $\check{X} \in \mathcal{X}$ such that $X + \check{X} \in \mathcal{X}$. Acceptability requires the shortfall of a trading strategy to be bounded by a maximal wealth process, rather than a constant, as in the case of the admissibility requirements.

1.1.5. *The utility-maximization problem.* Starting with initial wealth x and q^j units of each of the non-traded assets f^j in the portfolio, an economic agent can invest in the market and achieve any of the wealths in the collection

$$\mathcal{X}(x,q) := \{ X \equiv x + H \cdot S \mid X \text{ is acceptable and } X_T + \langle q, f \rangle \geq 0 \}.$$

where $q \equiv (q^1, \dots, q^N)$ and $\langle \cdot, \cdot \rangle$ denotes inner product in the Euclidean space $\mathbb{R}^N$ (if $N = 0$, the variable q is absent). The agent's goal is to choose $X \in \mathcal{X}(x,q)$ in such a way as to maximize $\mathbb{E}^{\mathbb{P}}[U(X_T + \langle q, f \rangle)]$, where $\mathbb{E}^{\mathbb{P}}$ is used to denote expectation under $\mathbb{P}$, and the utility U is a function mapping $(0,\infty)$ into $\mathbb{R}$, which is strictly increasing and strictly concave, continuously differentiable and satisfies the Inada conditions: $U'(0+) = \infty$, $U'(\infty) = 0$. The above *utility maximization*

problem is considered for all $(x,q) \in \mathcal{K}$, where $\mathcal{K}$ is the *interior* of the convex cone $\{(x,q) \mid \mathcal{X}(x,q) \neq \emptyset\} \subseteq \mathbb{R}^{N+1}$. In the liquid case $N=0$, (NFLVR) implies that $\mathcal{K} = (0,\infty) = \mathrm{Int}[0,\infty)$. In the general case, its geometry depends on the interplay of the liquid and illiquid assets. In Lemma 1 of [13] it is shown that the assumption (S-REP) of sub- and super- replicability of the illiquid assets is equivalent to $(x,0) \in \mathcal{K}$, for all $x > 0$ (and always, trivially, satisfied when $N=0$).

It is useful to consider the *value function* or *indirect utility* of this problem as a function of both the initial wealth x that can be distributed in the liquid assets, and the positions $q \in \mathbb{R}^N$ held in the illiquid assets; thus, we define the *indirect utility*

$$u(x,q) := \sup_{X \in \mathcal{X}(x,q)} \mathbb{E}^{\mathbb{P}}[U(X_T + \langle q, f\rangle)] \tag{1.1}$$

for $(x,q) \in \mathcal{K}$. The specification of the indirect utility as a function of both the initial capital and the holdings in the illiquid assets is convenient if one wants to introduce utility-based prices.

1.1.6. *Marginal utility-based prices.* For an agent with an initial wealth x and an initial position q in $N \geq 1$ illiquid assets, a *marginal utility-based price* for $f = (f^1, \dots, f^N)$ is a vector $p \equiv p(f;x,q) \in \mathbb{R}^N$ such that if f were liquid and traded at prices p, the utility-maximizing agent would be indifferent to changing his/her positions in f. In more concrete terms, we must have $u(x,q) \;\geq\; u(\tilde{x},\tilde{q})$, for all $(\tilde{x},\tilde{q}) \in \mathcal{K}$ with $x + \langle q,p\rangle = \tilde{x} + \langle \tilde{q}, p\rangle$. In [12], the authors have shown that marginal utility-based prices always exist, but do not, surprisingly, have to be unique. More precisely, the *set* of marginal utility-based prices for f (with initial positions x and q) is

$$\mathcal{P}(f;x,q;U) \;:=\; \{y^{-1}r \mid (y,r) \in \partial u(x,q)\}, \tag{1.2}$$

where $\partial u(x,q)$ is the superdifferential of the concave function u at $(x,q) \in \mathcal{K}$.

1.1.7. *The dual problem.* In order to solve the *primal* (utility maximization) problem, it is useful to consider the related *dual problem*

$$v(y,r) := \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}^{\mathbb{P}}[V(Y_T)], \tag{1.3}$$

where $V(y) := \sup_{x>0}\{U(x) - xy\}$ is the Legendre-Fenchel transform of $U(\cdot)$ and $\mathcal{Y}(y,r)$ is defined to be the class of all non-negative càdlàg processes Y such that $Y_0 = y$, YX is a supermartingale for all $X \in \mathcal{X}$ and such that $\mathbb{E}[Y_T(X_T + \langle q,f\rangle)] \leq xy + \langle q,r\rangle$ holds for all $(x,q) \in \mathcal{K}$ and $X \in \mathcal{X}(x,q)$. The obvious simplifications apply when $N=0$. The dual problem (1.3) is defined for all $(y,r) \in \mathcal{L}$, where we set $\mathcal{L} := \mathrm{ri}(-\mathcal{K})^\circ$, with $(-\mathcal{K})^\circ = \big\{(y,r) \in \mathbb{R}^{N+1} \mid xy + \langle q,r\rangle \;\geq 0,\ \forall\,(x,q) \in \mathcal{K}\big\}$. In words, $\mathcal{L}$ is the *relative interior* of the *polar cone* $(-\mathcal{K})^\circ$ of $-\mathcal{K}$. We have the set equality

$$\mathcal{P}(f) = \{p \in \mathbb{R}^N \mid (1,p) \in \mathcal{L}\} \tag{1.4}$$

(see equation (9), p. 850 in [13]) with $\mathcal{P}(f)$ defined in (N-TRAD) to be the set of *arbitrage-free* prices for f. For future reference, for any $p \in \mathcal{P}(f)$ we set

$$\mathcal{Q}'(p) := \{\mathbb{Q} \in \mathcal{Q}' \mid \mathbb{E}^{\mathbb{Q}}[f] = p\},$$

where $\mathbb{E}^{\mathbb{Q}}[f] := (\mathbb{E}^{\mathbb{Q}}[f^1], \dots, \mathbb{E}^{\mathbb{Q}}[f^N]) \in \mathbb{R}^N$ and $\mathcal{Q}'(p) = \mathcal{Q}'$ if $N = 0$. The authors of [13] show that $\mathcal{Q}'(p) \neq \emptyset$ for *all* $p \in \mathcal{P}$.

1.1.8. *A theorem of Huggonier and Kramkov.* We conclude this section by stating a version of the main theorem of [13], which will be referred to throughout the sequel.

Theorem 1.1 (Huggonier and Kramkov (2004))**.** *Suppose that $v(y, 0) < \infty$ for all $y > 0$. Then, the functions u and v are finitely valued on $\mathcal{K}$ and $\mathcal{L}$, respectively, and are conjugate to each other:*

$$v(y, r) = \sup_{(x,q) \in \mathcal{K}} \{u(x, q) - xy - \langle q, r\rangle\},$$

$$u(x, q) = \inf_{(y,r) \in \mathcal{L}} \{v(y, r) + xy + \langle q, r\rangle\}.$$

Furthermore, for each $(x, q) \in \mathcal{K}$ we have $\partial u(x, q) \subseteq \mathcal{L}$; actually,

$$\partial u(x, q) = \begin{cases} \{y\} \times R, & N \geq 1, \\ \{y\}, & N = 0 \end{cases} \tag{1.5}$$

for some $y = y(x, q) \in (0, \infty)$ and some compact and convex set $R = R(x, q) \subseteq \mathbb{R}^N$. The optimal solutions $\hat{X}(x, q)$ and $\hat{Y}(y, r)$ for the primal and dual problems exist for all $(x, q) \in \mathcal{K}$ and $(y, r) \in \mathcal{L}$. Moreover, if $(y, r) \in \partial u(x, q)$, we have the $\mathbb{P}$-a.s. equality $\hat{Y}_T(y, r) = U'(\hat{X}_T(x, q) + \langle q, f\rangle)$

Remark 1.2. For all $(x, q) \in \mathcal{K}$, the superdifferential $\partial u(x, q)$ is a compact and convex subset of $\mathcal{L}$. Moreover, since $y > 0$ for $(y, r) \in \partial u(x, q)$, equation (1.2) and the set-equality (1.4) imply that $\mathcal{P}(f; x, q; U)$ is a convex and compact subset of $\mathcal{P}(f)$ — in other words, marginal utility-based prices are arbitrage-free prices.

We note some further properties of the utility and value functions above. It follows from the properties of convex conjugation that the function V is strictly convex, continuously differentiable and strictly decreasing on its natural domain. The value functions u and v are conjugates of each other — $u(\cdot, q)$ is strictly concave, strictly increasing, while $v(\cdot, q)$ is strictly convex and strictly decreasing. Both $u(\cdot, q)$ and $v(\cdot, q)$ are continuously differentiable. Considered as functions of the second argument $u(x, \cdot)$ and $v(y, \cdot)$ are continuous on the interiors of effective domains.

1.2. **Stability analysis.** Having described the utility-maximization setting of [13], we turn to the central question of the present paper: *what are the consequences of model and/or preference misspecification for the optimal investment problem (as described in the previous section)?*

1.2.1. *Problem formulation.* In mathematical terms, we can ask whether the mapping that takes as inputs a utility function U and a probability measure $\mathbb{P}$ and produces the optimal wealth process and the set of utility-based prices for contingent claims (the illiquid assets) is continuous. Of course, appropriate topologies on the sets of the probability measures, utility functions, terminal wealth processes and prices need to be introduced.

Focusing on the special case of the logarithmic utility in a complete Itô-process market, the authors of [22] determine certain conditions on topologies governing the convergence of stock-price

processes, which are *necessary* for convergence in probability on the space of the terminal wealth processes in *all* models. A similar approach in our case obviates the need for, at least, the following set of assumptions:

i) the class of probability measures is endowed with the topology of convergence in *total variation*, and
ii) the space of utility functions is topologized by *pointwise* convergence.

Remark 1.3. In general, the topology of pointwise convergence lacks the operational property of metrizability. However, when restricted to a class of concave functions — such as utility functions — it becomes equivalent to the metrizable topology of uniform convergence on compact sets. From the economic point of view, such convergence is natural because — despite its apparent coarseness — it implies pointwise (and locally uniform) convergence of derivatives (marginal utilities), and thus, convergence in the local Sobolev space $W^{1,\infty}_{loc}$. It is implied, for example, by the convergence of (absolute or relative) risk aversions under the appropriate normalization. The pointwise convergence of utility functions is the most used notion of convergence for utility functions in economic literature (see [14] or [3] in the financial framework, or [1] for a more general discussion and relation to other, less used notions of convergence).

In the sequel, we consider two sequences $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$ of probability measures and utilities, together with the "limiting" probability measure $\mathbb{P}$ and utility function U. These will always be assumed to satisfy the following (equivalency and) convergence condition:

(CONV) $\quad \forall\, n\in\mathbb{N},\ \mathbb{P}_n\sim\mathbb{P},\ \lim_{n\to\infty}\mathbb{P}_n=\mathbb{P}$ in total variation and $\lim_{n\to\infty}U_n=U$ pointwise.

Remark 1.4. Some aspects of the approach of [22] can be recovered in our setting when the utility function U is kept constant and there are no illiquid assets ($N=0$). In order to see that, recall that in [22], the authors consider a general (right-continuous and complete) filtration $\mathbf{F}=(\mathcal{F}_t)_{t\in[0,T]}$, on which a one-dimensional continuous local martingale M is defined. They vary the model by considering a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of market-price-of-risk processes, giving rise to a sequence of stock-price processes

$$\mathrm{d}S^{\lambda_n}(t)=\lambda_n(t)\,\mathrm{d}\langle M\rangle(t)+\mathrm{d}M(t).$$

They study the convergence of the outputs of the utility-maximization problems in the sequence $(S^{\lambda_n})_{n\in\mathbb{N}}$ of models, while keeping the "physical" measure $\mathbb{P}$ fixed.

In our framework, we keep the functional representation of the models constant (as the same functions mapping Ω into the appropriate co-domain), but the measure $\mathbb{P}$ changes. To see the connection, let S be a continuous-path semimartingale. Then, $\mathrm{d}S(t)=\lambda(t)\,\mathrm{d}\langle M\rangle(t)+\mathrm{d}M(t)$, where M is a local $\mathbb{P}$-martingale. For $n\in\mathbb{N}$ let $\mathbb{P}_n\sim\mathbb{P}$; then, Girsanov's theorem enables us to write $\mathrm{d}S(t)=\lambda_n(t)\,\mathrm{d}\langle M\rangle(t)+\mathrm{d}M_n(t)$, where M_n is a local $\mathbb{P}_n$-martingale with $\langle M_n\rangle=\langle M\rangle$. It is straightforward to check that $\lim_{n\to\infty}\mathbb{P}_n=\mathbb{P}$ in total variation implies $\lim_{n\to\infty}\int_0^T\|\lambda_n(t)-\lambda(t)\|^2\mathrm{d}\langle M\rangle(t)=0$ in $\mathbf{L}^0$. Conversely, the latter convergence, coupled with requiring that M has the

predictable representation property with respect to the filtration $\mathbf{F}$ and some uniform integrability conditions, imply that $\lim_{n\to\infty} \mathbb{P}_n = \mathbb{P}$ in total variation.

Remark 1.5. The equivalence of all probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ to $\mathbb{P}$ as required by (CONV) is a rather strong condition — in particular, it pins down the quadratic variation of S and this means that model misspecifications with respect to volatility in simple Itô-process models cannot be dealt. Our choice to impose such a requirement nevertheless is based on the following two observations:

(1) Stability in the general (non-equivalent) case can only be studied in the distributional sense; equivalence allows one to talk about convergence in probability. Such problems do not arise when one only considers numerical objects, such as prices of contingent claims for example.
(2) The structure of the dual sets (the sets of equivalent martingale measures) in the limit and that in the pre-limit models differ greatly in typical non-equivalent cases. This puts a severe limitation on the applicability of our method.

In special cases, however, there exists a simple way around the equivalence assumption, based on the observation that the subject of importance is not the asset-price vector S itself, but the collection of all wealth processes that are to be used in the utility maximization problem. This simple observation allows, for example, treatment of stochastic volatility models. The example below illustrates the general principle of how the equivalence requirement can be "avoided":

Suppose that under $\mathbb{P}$ we have the dynamics $\mathrm{d}S^i_t/S^i_t =: \mathrm{d}R^i_t = \mu^i_t \mathrm{d}t + \sum_{j=1}^d \sigma^{ij}_t \mathrm{d}W^j$ for $i = 1, \dots, d$, where $W = (W^j)_{1\le j\le d}$ is an $\mathbf{F}$-Brownian motion, $\mu = (\mu^i)_{1\le i\le d}$ and $\sigma = (\sigma^{ij})_{1\le i\le d, 1\le j\le d}$ are $\mathbf{F}$-predictable and σ is assumed to be non-singular-valued. It follows that for the returns vector $R = (R^i)_{1\le i\le d}$ we can write $\mathrm{d}R_t = \sigma_t(\lambda_t \mathrm{d}t + \mathrm{d}W_t)$, where $\lambda := \sigma^{-1}\mu$ is the Sharpe ratio. The non-singularity of σ implies that the set of wealth processes obtained by trading in S is the same as the one obtained by trading in assets with returns given by $\widetilde{R} = (\widetilde{R}^i)_{1\le i\le d}$ satisfying $\mathrm{d}\widetilde{R}_t = \lambda_t \mathrm{d}t + \mathrm{d}W_t$. This trick allows to get rid of the dependance on σ.

Suppose now we want to check the effect of changing both σ and μ — for example we want to see what will happen if $(\mu^{(n)}, \sigma^{(n)})$ converge to (μ, σ) in some sense. Define $\lambda^{(n)} = (\sigma^{(n)})^{-1}\mu^{(n)}$ (assume that each $\sigma^{(n)}$ is non-singular-valued) and $\mathbb{P}_n$ via the density (assuming that the exponential local martingale below is uniformly integrable):

$$\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\Big(\int_0^T (\lambda^{(n)}_t - \lambda_t)\mathrm{d}W_t - \frac{1}{2}\int_0^T \|\lambda^{(n)}_t - \lambda_t\|^2 \mathrm{d}t \Big)$$

As long as $\lim_{n\to\infty} \int_0^T \|\lambda^{(n)}_t - \lambda_t\|^2 \mathrm{d}t = 0$ (in probability) we have $\lim_{n\to\infty} \mathbb{P}_n = \mathbb{P}$ in total variation. Define new return processes $R^{(n)}$ via $\mathrm{d}R^{(n)}_t = \sigma^{(n)}_t\big(\lambda^{(n)}_t \mathrm{d}t + \mathrm{d}W^{(n)}_t\big) = \mu^{(n)}\mathrm{d}t + \sigma^{(n)}_t \mathrm{d}W^{(n)}_t$, where $W^{(n)} := W - \int_0^\cdot (\lambda^{(n)}_t - \lambda_t)\mathrm{d}t$ is $\mathbb{P}_n$-Brownian motion. The induced set of wealth processes by investing in asset-prices with returns $R^{(n)}$ is the same as the one obtained if the asset-prices had returns $\widetilde{R}^{(n)}$ that satisfied $\mathrm{d}\widetilde{R}^{(n)}_t = \lambda^{(n)}_t \mathrm{d}t + \mathrm{d}W^{(n)}_t = \mathrm{d}\widetilde{R}_t$, which in turn is the same as the *original* set of wealth processes obtained by investing in S. In this indirect way, we can study changes of

both drift and volatility in the model, while keeping our framework of only changing the probability measure and not the asset prices.

1.2.2. *A uniform-integrability condition.* Unfortunately, the modes of convergence in (CONV) are not strong enough for stability: [22] contains a simple example. In the setting of their example, $T = 1$ and there exists one liquid asset S whose $\mathbb{P}$-dynamics ($\mathbb{P}$ being the "limiting measure") is given by $\mathrm{d}S_t = S_t \mathrm{d}W_t$. W is a $\mathbb{P}$-Brownian motion, and the filtration is the (augmentation of the) one generated by W. The sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$, of measures is defined via $\mathrm{d}\mathbb{P}_n/\mathrm{d}\mathbb{P} = \varphi_n(W_1)$, where $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of positive real functions with $\lim_{n\to\infty} \varphi_n = 1$, pointwise. The utility function involved — in their treatment only the model changes and the utility is fixed — is *unbounded from above* (and is, in fact, a simple power function). What the authors of [22] show is that convergence of the optimal wealth processes in probability might fail — convergence of $(\mathbb{P}_n)_{n\in\mathbb{N}}$ to $\mathbb{P}$ in total variation is simply not enough. Moreover, their choice of the functions φ_n is such that $\mathrm{d}\mathbb{P}_n/\mathrm{d}\mathbb{P} \to 1$ in $\mathbf{L}^2$, and a simple variation of their argument may be used to show that, in fact, the $\mathbf{L}^p$ convergence will not be universally sufficient, no matter how large $p \in (1,\infty)$ is chosen. The appropriate strengthening of the requirement (CONV), as shown by [22], is the replacement of the classical $\mathbf{L}^p$ spaces by the Orlicz spaces related to the utility function U. In the present setting, where the variation in the model, as well as in the utility function, has to be taken into account, such a replacement leads to the following condition (in which $V_n^+(x) := \max\{V_n(x), 0\}$):

$$\text{(UI)} \qquad \forall\, p \in \mathcal{P},\ \exists\, \mathbb{Q} \in \mathcal{Q}'(p),\ \forall\, y > 0,\quad \left(\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\, V_n^+\Big(y \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_n} \Big) \right)_{n\in\mathbb{N}} \text{ is } \mathbb{P}\text{-uniformly integrable.}$$

1.2.3. *On condition (UI).* The following special cases illustrate the meaning and restrictiveness of the condition (UI). The convergence requirement (CONV) is assumed throughout.

(1) It has been shown in [22] that (the appropriate version of) the condition (UI) is both sufficient *and necessary* in complete financial markets. In the incomplete case, and still in the setting of [22], it is "close to" being necessary — the gap arising because of the technical issues stemming from the fact that the dual minimizers do *not* have to be countably-additive measures.

(2) When there are no illiquid assets ($N = 0$), the set $\mathcal{P}$ has no meaning and any martingale measure $\mathbb{Q}$ can be used in (UI). Also, in the case when the market is complete, the set $\mathcal{P}$ is a singleton and the unique equivalent martingale measure $\mathbb{Q}$ has to be used in (UI).

(3) The (UI) condition is immediately satisfied if the sequence $(U_n)_{n\in\mathbb{N}}$ is uniformly bounded from above. Indeed, in that case we have $\sup_{n\in\mathbb{N}} V_n^+ \le C$ for some $C > 0$ (the uniform upper bound on the utilities) and the sequence $(\mathrm{d}\mathbb{P}_n/\mathrm{d}\mathbb{P})_{n\in\mathbb{N}}$ is $\mathbb{P}$-uniformly integrable in view of its $\mathbf{L}^1(\mathbb{P})$ convergence.

(4) If the previous example corresponds to the duality between $\mathbf{L}^\infty$ and $\mathbf{L}^1$, the present one deals with the case of $\mathbf{L}^{\hat{p}}$ and $\mathbf{L}^{\hat{q}}$, $\hat{p}^{-1} + \hat{q}^{-1} = 1$. Indeed, assume the following conditions:

(a) there exist constants $c > 0$, $d \in \mathbb{R}$ and $0 < \alpha < 1$ (the case $\alpha = 0$ corresponds to the logarithmic function, and can be treated in a similar fashion) such that $U_n(x) \leq cx^\alpha + d$, for all $n \in \mathbb{N}$,

(b) the sequence $(\mathrm{d}\mathbb{P}_n/\mathrm{d}\mathbb{P})_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^{\hat{p}}$, for some $\hat{p} > (1-\alpha)^{-1}$, and

(c) for each $p \in \mathcal{P}$ there exists $\mathbb{Q}_p \in \mathcal{Q}'(p)$ such that $(\mathrm{d}\mathbb{Q}_p/\mathrm{d}\mathbb{P})^{-1} \in \mathbf{L}^{\hat{q}}$, where we set $\hat{q} := \frac{\hat{p}\alpha}{\hat{p}(1-\alpha)-1}$. Note that this requirement is not as strong as it seems, as it is closely related to the finiteness in the dual problem.

Then, for all $y > 0$,

$$V_n(y) = \sup_{x>0}\{U_n(x) - xy\} \leq \sup_{x>0}[cx^\alpha + d - xy] \leq Cy^{-\frac{\alpha}{1-\alpha}} + D, \tag{1.6}$$

where $C, D \in \mathbb{R}$ are positive constants. For arbitrary but fixed $p \in \mathcal{P}$ and $y > 0$ define $\gamma := \hat{q}\hat{p}(1-\alpha)(\hat{q}+\hat{p}\alpha)^{-1}$ so that $1 < \gamma < \hat{p}$, where $\hat{q} > 0$ has been defined above. Hölder's inequality (applied in the last inequality below) and the estimate (1.6) imply that

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\, V_n^+\Big(y\frac{\mathrm{d}\mathbb{Q}_p}{\mathrm{d}\mathbb{P}_n}\Big)\right)^\gamma\right] &\leq \mathbb{E}\left[\left(C\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\left(y\frac{\mathrm{d}\mathbb{Q}_p}{\mathrm{d}\mathbb{P}_n}\right)^{-\frac{\alpha}{1-\alpha}} + D\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\right)^\gamma\right] \\
&\leq 2^{\gamma-1}C^\gamma y^{-\frac{\alpha}{1-\alpha}}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{\mathrm{d}\mathbb{Q}_p}{\mathrm{d}\mathbb{P}}\right)^{-\frac{\gamma\alpha}{1-\alpha}}\right] + 2^{\gamma-1}D^\gamma\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\right)^\gamma\right] \\
&\leq 2^{\gamma-1}C^\gamma y^{-\frac{\alpha}{1-\alpha}}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\right)^{\hat{p}}\right]^{\frac{\gamma}{\hat{p}(1-\alpha)}}\ \mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}_p}{\mathrm{d}\mathbb{P}}\right)^{-\hat{q}}\right]^{1-\frac{\gamma}{\hat{p}(1-\alpha)}} \\
&+ 2^{\gamma-1}D^\gamma\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}\right)^\gamma\right],
\end{aligned}$$

which implies (UI).

(5) A family $(U_n)_{n\in\mathbb{N}}$ of utility functions is said to have a *uniform reasonable asymptotic elasticity*, if there exist constants $x_0 > 0$ and $\delta < 1$ such that $xU_n'(x) \leq \delta U_n(x)$ for all $x > x_0$ and $n \in \mathbb{N}$. Then, one can show (see Proposition 6.3 in [17]) that for each fixed $y > 0$, there exist $k, l > 0$ such that $V_n^+(yz) \leq kV_n^+(z) + l$ for all $z > 0, n \in \mathbb{N}$. In other words, under uniform reasonable asymptotic elasticity the "annoying" universal quantification over all $y > 0$ in (UI) can be left out — considering only the case $y = 1$ is enough.

(6) Several other sufficient conditions for (UI) in the case when $V_n = V$ for all $n \in \mathbb{N}$ and $N = 0$ are given in [22].

1.2.4. *The main result.* The statement of our main result, whose proof will be the given in Section 2 below, follows. In order to keep the unified notation for the cases $N = 0$ and $N > 0$, we introduce the following conventions (holding throughout the remainder of the paper): all the statements in the sequel will notationally correspond to the case $N > 0$, and should be construed literally in that case. When $N = 0$, the arguments r should be understood to take values in the one-element set $\mathbb{R}^0$, which we identify with $\{0\}$. Similarly, the variables p and q will take the value 0, and $\mathcal{Q}(0) = \mathcal{Q}'$. In this case, a pair such as $(x, 0)$ will be identified with the constant $x \in \mathbb{R}$.

Theorem 1.6. *Assume that* (NFLVR) *and* (S-REP) *are in force, and consider a sequence* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *of probability measures and a sequence* $(U_n)_{n\in\mathbb{N}}$ *of utility functions such that* (CONV) *and* (UI) *hold. Furthermore, let* $(x_n, q_n)_{n\in\mathbb{N}}$ *be a* $\mathcal{K}$*-valued sequence with* $\lim_{n\to\infty}(x_n, q_n) =: (x, q) \in \mathcal{K}$ *and* $(y_n, r_n)_{n\in\mathbb{N}}$ *an* $\mathcal{L}$*-valued sequence with* $\lim_{n\to\infty}(y_n, r_n) =: (y, r) \in \mathcal{L}$.

Set $u_n = u(x_n, q_n; U_n, \mathbb{P}_n)$, $u_\infty = u(x, q; U, \mathbb{P})$, $v_n = v(y_n, r_n; U_n, \mathbb{P}_n)$, $v_\infty = v(y, r; U, \mathbb{P})$, *and let* $\frac{\partial}{\partial x}u_n$, $\frac{\partial}{\partial x}u_\infty$, $\frac{\partial}{\partial y}v_n$, $\frac{\partial}{\partial y}v_\infty$, *be the corresponding derivatives with respect to the first variable. Similarly, set* $\hat{X}_n = \hat{X}_T(x_n, q_n; U_n, \mathbb{P}_n)$, $\hat{X}_\infty = \hat{X}_T(x, q; U, \mathbb{P})$, $\hat{Y}_n = \hat{Y}_T(y_n, r_n; V_n, \mathbb{P}_n)$ *and* $\hat{Y}_\infty = \hat{Y}_T(y, r; V, \mathbb{P})$. *Finally, set* $\mathcal{P}_n = \mathcal{P}(f; x_n, q_n; U_n, \mathbb{P}_n)$ *and* $\mathcal{P}_\infty = \mathcal{P}(f; x, q; U, \mathbb{P})$.

Then, we have the following limiting relationships for the value functions and the optimal solutions in the primal and dual problems:

(1) $\lim_{n\to\infty} u_n = u_\infty$, $\lim_{n\to\infty} v_n = v_\infty$, $\lim_{n\to\infty} \frac{\partial}{\partial x}u_n = \frac{\partial}{\partial x}u_\infty$, *and* $\lim_{n\to\infty} \frac{\partial}{\partial y}v_n = \frac{\partial}{\partial y}v_\infty$.

(2) $\lim_{n\to\infty} \hat{X}_n = \hat{X}_\infty$ *and* $\lim_{n\to\infty} \hat{Y}_n = \hat{Y}_\infty$ *in* $\mathbf{L}^0$, *where, as usual,* $\mathbf{L}^0$ *is the family of all random variables endowed with the topology of convergence in probability.*

(3) *for all* $\epsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\mathcal{P}_n \subseteq \mathcal{P}_\infty + \epsilon B^N, \text{ for } n \geq n_0,$$

where B^N *is the open ball of unit radius in* $\mathbb{R}^N$.

Remark 1.7. The set-inclusion $\mathcal{P}_n \subseteq \mathcal{P}_\infty + \epsilon B^N$ for all n large enough is an *upper hemicontinuity*-type property of the correspondence of marginal utility-based prices. It says that all possible limit points of all possible sequences of marginal utility-based prices will belong to the limiting price-set. It does *not* imply that this last set will be equal to the set of all these possible limit points — indeed, it might be strictly larger.

2. Proofs

This section concentrates on the proof of our main Theorem 1.6. First, we prove a lower semicontinuity-type result for of the dual value function, which, interestingly, does *not* depend on the assumption (UI) from subsection 2.2. Then, in subsection 2.3, we use both (CONV) and (UI) to establish a complementary upper semicontinuity-type property for the dual value function. Continuity of the primal value function and upper hemicontinuity of the correspondence of marginal utility-based prices are proved in subsection 2.4. Finally, subsection 2.5 deals with convergence in $\mathbf{L}^0$ of the dual optimal element. Convergence in $\mathbf{L}^0$ of the optimal terminal wealths is then established using the continuity of the value functions.

2.1. **Preliminary remarks.** We start by making some remarks and conventions that will be in force throughout the proof. Since there are many different probability measures floating around, we choose $\mathbb{P}$ to serve as the baseline: *all* expectations $\mathbb{E}$ in the sequel will be taken with respect to the probability $\mathbb{P}$ — we then consider the Radon-Nikodym densities $Z_n := d\mathbb{P}_n/d\mathbb{P}$ and use them whenever we want to take expectation with respect to some $\mathbb{P}_n$. The space $\mathbf{L}^0$ of all a.s.-finite random variables is the same for all (equivalent) probabilities and thus requires no identifier. The

notation $\mathbf{L}^1$ is reserved for $\mathbf{L}^1(\mathbb{P})$. Observe that the convergence $\lim_{n\to\infty}\mathbb{P}_n = \mathbb{P}$ in total variation is equivalent to the convergence $\lim_{n\to\infty} Z_n = 1$, in $\mathbf{L}^1$. By Scheffe's Lemma (see [27], p. 55) this is equivalent to the (seemingly weaker) statement $\lim_{n\to\infty} Z_n = 1$ in $\mathbf{L}^0$.

Let us move on to the discussion of utility functions. Note that pointwise (and thus, by concavity, uniform on compacts) convergence of the sequence $(U_n)_{n\in\mathbb{N}}$ to a utility U will imply pointwise convergence of the sequence of Legendre-Fenchel transforms $(V_n)_{n\in\mathbb{N}}$ to the corresponding Legendre-Fenchel transform V of the limiting utility U. We actually get a lot more: the sequences $(U_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}}$ as well as their derivatives $(U_n')_{n\in\mathbb{N}}$, $(V_n')_{n\in\mathbb{N}}$ converge *uniformly* on compact subsets of $(0,\infty)$ to their respective limits U, V, U' and V' (see, e.g., [25] for a general statement or [22] for a simple self-contained proof of this result). A multidimensional version of this result will be used later on in subsection 2.5.

Also, without loss of generality we assume that each of the utility functions involved here is normalized in such a way as to have $U_n(1) = 0$ and $U_n'(1) = 1$ — this will mean that $V_n(1) = V_n'(1) = -1$. One can check that nothing changes in the validity of our Theorem 1.6 if we make this simple affine transformation in the utilities, but the proofs below will be much cleaner. Indeed, we can define a new sequence $(\hat{U}_n)_{n\in\mathbb{N}}$ via

$$\hat{U}_n(x) = \frac{U_n(x) - U_n(1)}{U_n'(1)};$$

pointwise convergence of the original sequence $(U_n)_{n\in\mathbb{N}}$ implies pointwise convergence of both $(\hat{U}_n)_{n\in\mathbb{N}}$ and $(\hat{V}_n)_{n\in\mathbb{N}}$.

2.2. **A lower semicontinuity-type property of the dual value function.** We assume that (NFLVR), (S-REP) and (CONV) hold throughout this subsection. The assumption (UI) is not yet needed.

2.2.1. *Preparatory work.* Notice that $(Z_n)_{n\in\mathbb{N}}$ is $\mathbb{P}$-uniformly integrable, and, more generally, that the *convex hull* $\mathrm{conv}(Z_n; n\in\mathbb{N})$ is $\mathbb{P}$-uniformly integrable, as well. Observe that

$$v_n(y,r) \equiv v(y,r;\ U_n,\mathbb{P}_n) = \inf_{g\in\mathcal{D}(y,r)} \mathbb{E}[Z_n V(g/Z_n)] = \mathbb{E}[Z_n V(g_n/Z_n)], \tag{2.1}$$

where

$$\mathcal{D}(y,r) := \{g\in\mathbf{L}^0 \mid 0\le g\le Y_T \text{ for some } Y\in\mathcal{Y}(y,r)\}, \tag{2.2}$$

$\mathcal{Y}(y,r)$ is the class of supermartingale deflators corresponding to the limiting probability measure $\mathbb{P}$, and $g_n\in\mathcal{D}(y,r)$ attains the infimum in (2.1). We wish to show that

$$v(y,r) \ \le\ \liminf_{n\to\infty} v_n(y,r) \tag{2.3}$$

The "liminf" in (2.3) we can be safely regarded as an *actual* limit, passing to an attaining subsequence if necessary. By the same token, we can also assume that the convergence $\lim_{n\to\infty} Z_n = 1$ holds almost surely, and not only in $\mathbf{L}^0$.

Lemma A.1.1 from [6] provides us with a finite random variable $h \geq 0$, and a sequence $(h_n)_{n\in\mathbb{N}}$ such that

$$\forall\, n \in \mathbb{N},\ h_n \in \operatorname{conv}(g_n, g_{n+1}, \dots) \text{ and } \lim_{n\to\infty} h_n = h, \text{ a.s.} \tag{2.4}$$

In [13] the authors show that for all $(y,r) \in \mathcal{L}$ the convex set $\mathcal{D}(y,r)$ is closed in $\mathbf{L}^0$, thus we have $h \in \mathcal{D}(y,r)$. For concreteness, let us write $h_n = \sum_{k=n}^{m_n} \alpha_k^n g_k$ for some $m_n \geq n$ and $0 \leq \alpha_k^n \leq 1$ such that $\sum_{k=n}^{m_n} \alpha_k^n = 1$. We then also set $\zeta_n := \sum_{k=n}^{m_n} \alpha_k^n Z_k$ and observe that $\lim_{n\to\infty} \zeta_n = 1$ holds almost surely; here, it is crucial that we have $\lim_{n\to\infty} Z_n = 1$ almost surely and not only in $\mathbf{L}^0$ — $\mathbb{R}$ is a locally convex space, while $\mathbf{L}^0$ is not.

2.2.2. *On the sequence* $(V_n)_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$ and $\epsilon \in (0,1)$, define the function V_n^ϵ as follows: set $V_n^\epsilon(x) = V_n(x)$ for $x \geq \epsilon$, and extend V_n^ϵ to $[0,\epsilon)$ in an affine and continuously differentiable way, i.e. match the zeroth and first derivatives at $x = \epsilon$. This recipe uniquely determines a decreasing and convex function V_n^ϵ. Of course, $\lim_{\epsilon\downarrow 0} \uparrow V_n^\epsilon(x) = V_n(x)$, for all $x > 0$. In the same manner as above, and using the function V, define V^ϵ for all $\epsilon \in (0,1)$.

Since $\lim_{n\to\infty} V_n(x) = V(x)$ and $\lim_{n\to\infty} V_n'(x) = V'(x)$ *uniformly* for $x \in [\epsilon, 1]$, we have that $\lim_{n\to\infty} V_n^\epsilon(x) = V^\epsilon(x)$, *uniformly* for $x \in [0,1]$. Notice that this uniform convergence fails in general for $\epsilon = 0$, unless $(U_n)_{n\in\mathbb{N}}$ is uniformly bounded from above (equivalently, if $(V_n)_{n\in\mathbb{N}}$ is uniformly bounded from above). It follows that

$$\forall\, \epsilon \in (0,1)\ \exists\, n_1(\epsilon) \in \mathbb{N},\ \forall\, x \in [0,1],\ \forall\, n \geq n_1(\epsilon),\ V_n^\epsilon(x) \geq V^\epsilon(x) - \epsilon. \tag{2.5}$$

Let $\tilde{V}_n$ denote the *convex minor* of the family $\{V_n, V_{n+1}, \dots\}$; that is, $\tilde{V}_n$ is the largest convex function that is dominated by all V_k for $k \geq n$. Each $\tilde{V}_n$ is clearly convex and decreasing. Observe also that $\tilde{V}_n(x) \geq -x$. Indeed, remembering that $V_n(1) = V_n'(1) = -1$, for all $n \in \mathbb{N}$, one concludes that $V_n(x) \geq -x$ for all $n \in \mathbb{N}$. In fact,

$$-1 + \int_1^x \Big(\inf_{k\geq n} V_k'(u) \Big) \mathrm{d}u \ \leq\ \tilde{V}_n(x)\ \leq\ V(x).$$

This last expression, and the fact that $\lim_{n\to\infty} V_n' = V'$ uniformly on compact subsets of $(0,\infty)$ imply that $\lim_{n\to\infty} \uparrow \tilde{V}_n = V$ uniformly on compact subsets of $(0,\infty)$.

Define now the "average" functions $\overline{V}_n(x) := \tilde{V}_n(x)/x$, $x > 0$ for all $n \in \mathbb{N}$, as well as $\overline{V}(x) := V(x)/x$, $x > 0$. Observe that $\lim_{n\to\infty} \uparrow \overline{V}_n = \overline{V}$ (increasing limit). The following, stronger, statement holds as well.

Lemma 2.1. $\lim_{n\to\infty} \overline{V}_n = \overline{V}$, *uniformly on* $[1,\infty)$.

Proof. We base the proof on Dini's theorem. In order to be able to use it we have to ensure that the sequence $\overline{V}_n(\infty)$ increases and converges to $\overline{V}(\infty)$, where $\overline{V}_n(\infty) := \lim_{x\to\infty} \overline{V}_n(x)$ and $\overline{V}(\infty) := \lim_{x\to\infty} \overline{V}(x) = \lim_{x\to\infty} V'(x) = 0$ (since V is the convex conjugate of $-U(-\cdot)$).

The normalization $V_n(1) = V_n'(1) = -1$ implies that $\tilde{V}_n(1) = \tilde{V}_n'(1) = -1$ and $V(1) = V'(1) = -1$, and that $\overline{V}$ and all $\overline{V}_n$ are *increasing* for $x \in [1,\infty)$.

Then, for an arbitrary $\delta > 0$, pick $M > 1$ so that $\overline{V}(M) > -\delta/2$ and $n_2 \equiv n_2(\delta, M) \in \mathbb{N}$ so that $\overline{V}_n(M) > \overline{V}(M) - \delta/2 > -\delta$ for all $n \geq n_2$. It follows that $\overline{V}_n(\infty) \geq \overline{V}_n(M) > -\delta$ for all $n \geq n_2$ and thus that $\lim_{n\to\infty} \overline{V}_n(\infty) = \overline{V}(\infty) = 0$. As proclaimed, Dini's theorem will imply that $\lim_{n\to\infty} \overline{V}_n = \overline{V}$ *uniformly* on $[1, \infty)$. □

Lemma 2.2. *The mapping $(z, y) \mapsto zV^\epsilon(y/z)$ is convex in $(z, y) \in (0, \infty)^2$. Furthermore, for each $\epsilon > 0$, there exists $n_0(\epsilon) \in \mathbb{N}$ such that for all $n \geq n_0(\epsilon)$ we have*

$$zV_n^\epsilon(y/z) \geq zV^\epsilon(y/z) - \epsilon(y + z).$$

for all pairs $(z, y) \in (0, \infty)^2$.

Proof. The fact that $(z, y) \mapsto zV^\epsilon(y/z)$ is convex in $(z, y) \in (0, \infty)^2$ is a consequence of the convexity of V^ϵ and is quite standard. A detailed proof can be found, for example, in [10], page 90.

For the second claim, pick $\epsilon > 0$, and use Lemma 2.1 to find a natural number $n_3(\epsilon)$ such that $\overline{V}_n^\epsilon(x) \geq \overline{V}^\epsilon(x) - \epsilon$ for *all* $x \geq 1$ and $n \geq n_3(\epsilon)$. Then, pick $n_1(\epsilon)$ as in (2.5). Finally, choose $n_0(\epsilon) := \max\{n_1(\epsilon), n_3(\epsilon)\}$. For all $n \geq n_0(\epsilon)$ we now have:

$$zV_n^\epsilon(y/z) \geq z(V^\epsilon(y/z) - \epsilon)\mathbb{I}_{\{y \leq z\}} + y(\overline{V}(y/z) - \epsilon)\mathbb{I}_{\{y > z\}} \geq zV^\epsilon(y/z) - \epsilon(y + z).$$

□

2.2.3. *The conclusion of the proof of (2.3).* Let $\epsilon \in (0, 1)$ be fixed, but arbitrary. According to Lemma 2.2, for all $n \geq n_0(\epsilon)$ we have $Z_n V_n^\epsilon(g_n/Z_n) \geq Z_n V^\epsilon(g_n/Z_n) - \epsilon(Z_n + g_n)$; applying expectation with respect to $\mathbb{P}$ and taking limits (remember that we have passed in a subsequence so that $\lim_{n\to\infty} v_n(y, r)$ exists) we get

$$\lim_{n\to\infty} v_n(y, r) \geq \limsup_{n\to\infty} \mathbb{E}[Z_n V_n^\epsilon(g_n/Z_n)] \geq \limsup_{n\to\infty} \mathbb{E}[Z_n V^\epsilon(g_n/Z_n)] - (y + 1)\epsilon. \tag{2.6}$$

Apply Lemma 2.2 again to get $\zeta_n V^\epsilon(h_n/\zeta_n) \leq \sum_{k=n}^{m_n} \alpha_k^n Z_k V^\epsilon(g_k/Z_k)$, where the sequence $(h_n)_{n\in\mathbb{N}}$ is the one of (2.4); this implies that

$$\limsup_{n\to\infty} \mathbb{E}[Z_n V^\epsilon(g_n/Z_n)] \;\geq\; \limsup_{n\to\infty} \sum_{k=n}^{m_n} \alpha_k^n \mathbb{E}[Z_k V^\epsilon(g_k/Z_k)] \;\geq\; \limsup_{n\to\infty} \mathbb{E}[\zeta_n V^\epsilon(h_n/\zeta_n)].$$

A combination of this inequality with the estimate (2.6) yields that

$$\lim_{n\to\infty} v_n(y, r) \;\geq\; \limsup_{n\to\infty} \mathbb{E}[\zeta_n V^\epsilon(h_n/\zeta_n)] - (y + 1)\epsilon. \tag{2.7}$$

Since $\overline{V}^\epsilon$ is increasing on $[1, \infty)$ and satisfies $\overline{V}^\epsilon(1) = -1$ and $\overline{V}^\epsilon(\infty) = 0$, one can choose $M > 1$ such that $\overline{V}^\epsilon(M) = -\epsilon$ and define $\overline{V}^{\epsilon,M}$ by requiring $\overline{V}^{\epsilon,M}(x) = \overline{V}^\epsilon(x)$ for $0 < x \leq M$, $\overline{V}^{\epsilon,M}(x) = 0$ for all $x \geq M + 1$, and interpolating in a continuous way between M and $M + 1$ so that $\overline{V}^\epsilon \leq \overline{V}^{\epsilon,M}$. Then $\overline{V}^{\epsilon,M} - \epsilon \leq \overline{V}^\epsilon \leq \overline{V}^{\epsilon,M}$ and

$$\zeta_n V^\epsilon(h_n/\zeta_n) = h_n \overline{V}^\epsilon(h_n/\zeta_n) \geq h_n \overline{V}^{\epsilon,M}(h_n/\zeta_n) - \epsilon h_n. \tag{2.8}$$

Observe that $h_n\overline{V}^{\epsilon,M}(h_n/\zeta_n) \le V^\epsilon(0)\zeta_n$; also, since $\overline{V}^{\epsilon,M}(x) \ge -1$ for all $x>0$ and $\overline{V}^{\epsilon,M}(x)=0$ for $x > M+1$, we have

$$h_n\overline{V}^{\epsilon,M}(h_n/\zeta_n) = h_n\overline{V}^{\epsilon,M}(h_n/\zeta_n)\mathbb{I}_{\{h_n \le (M+1)\zeta_n\}} \ge -(M+1)\zeta_n.$$

It follows that $|h_n\overline{V}^{\epsilon,M}(h_n/\zeta_n)| \le \kappa^{\epsilon,M}\zeta_n$, where $\kappa^{\epsilon,M} := \max\{V^\epsilon(0), M+1\}$. The sequence $(\zeta_n)_{n\in\mathbb{N}}$ is $\mathbb{P}$-uniformly integrable with $\lim_{n\to\infty}\zeta_n = 1$ a.s, and $\lim_{n\to\infty} h_n = h$, a.s. So, by (2.8), we have

$$\limsup_{n\to\infty} \mathbb{E}[\zeta_n V^\epsilon(h_n/\zeta_n)] \;\ge\; \mathbb{E}[h\overline{V}^{\epsilon,M}(h)] - \epsilon y \;\ge\; \mathbb{E}[h\overline{V}^{\epsilon}(h)] - \epsilon y \;=\; \mathbb{E}[V^\epsilon(h)] - \epsilon y.$$

Combining this last estimate with (2.7) we get

$$\lim_{n\to\infty} v_n(y,r) \;\ge\; \limsup_{n\to\infty} \mathbb{E}[\zeta_n V^\epsilon(h_n/\zeta_n)] - (y+1)\epsilon \;\ge\; \mathbb{E}[V^\epsilon(h)] - (2y+1)\epsilon$$

Now, since $V^\epsilon(h) \ge -h$ and $h \in \mathbf{L}^1$, one can use the monotone convergence theorem in the last inequality and the fact that $h \in \mathcal{D}(y,r)$ to get (as $\epsilon \downarrow 0$) that

$$\lim_{n\to\infty} v_n(y,r) \;\ge\; \mathbb{E}^{\mathbb{P}}[V(h)] \;\ge\; v(y,r),$$

which finishes the proof.

2.3. **Limiting behavior of the sequence of dual value functions.** From now on, we assume that all four conditions (NFLVR), (S-REP), (CONV) and (UI) hold. The first order of business is to study the behavior of the limit superior of the sequence of the dual value functions. Then, we combine the obtained result with that of subsection 2.2.

2.3.1. *Auxiliary Results.* For future reference, for any $p \in \mathcal{P}$, $\mathbb{Q} \in \mathcal{Q}'(p)$ and $(y,r) \in \mathcal{L}$ such that $yp = r$ we define

$$\mathcal{B}(y,r,\mathbb{Q}) := \{g \in \mathcal{D}(y,r) \mid \tfrac{1}{g}\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathbf{L}^\infty.\}. \tag{2.9}$$

Since $\mathcal{D}(y,r)$ is convex and $y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{D}(y,r)$, we have that for all $g \in \mathcal{D}(y,r)$ and $k \in \mathbb{N}$, $k^{-1}(y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}) + (1-k^{-1})g \in \mathcal{B}(y,r,\mathbb{Q})$; in particular, $\mathcal{B}(y,r,\mathbb{Q}) \neq \emptyset$.

Lemma 2.3. *Fix $y>0$ and $p \in \mathcal{P}$, and let $\mathbb{Q} \in \mathcal{Q}'(p)$ be such that $V^+(y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}) \in \mathbf{L}^1(\mathbb{P})$. Then, with $r := yp$ and $\mathcal{B}(y,r,\mathbb{Q})$ defined in* (2.9)*, we have*

$$v(y,r) = \inf_{g \in \mathcal{B}(y,r,\mathbb{Q})} \mathbb{E}[V(g)].$$

Proof. Let $g_* \in \mathcal{D}(y,r)$ satisfy $v(y,r) = \mathbb{E}[V(g_*)]$. For all $k \in \mathbb{N}$ define $g_{*,k} := (1-k^{-1})g_* + k^{-1}(y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})$. Then, $g_{*,k} \in \mathcal{B}(y,r,\mathbb{Q})$ and $\mathbb{E}[V(g_{*,k})] \le (1-k^{-1})\mathbb{E}[V(g_*)] + k^{-1}\mathbb{E}[V(y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})]$. Finally, since $V(y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}) \in \mathbf{L}^1$, we get that $\lim_{k\to\infty}\mathbb{E}[V(g_{*,k})] = v(y,r)$. □

Lemma 2.4. *Suppose that for some $f \in \mathbf{L}^0_+$ the collection $(Z_nV_n^+(f/Z_n))_{n\in\mathbb{N}}$ of random variables is $\mathbb{P}$-uniformly integrable. Let also $g \in \mathbf{L}^1(\mathbb{P})$ be such that $g \ge f$ almost surely. Then, $\lim_{n\to\infty} Z_nV_n(g/Z_n) = V(g)$ in $\mathbf{L}^1(\mathbb{P})$.*

Proof. Since we have $\lim_{n\to\infty} Z_n V_n(g/Z_n) = V(g)$ in $\mathbf{L}^0$, we only have to show that the collection $(Z_n V_n(g/Z_n))_{n\in\mathbb{N}}$ of random variables is $\mathbb{P}$-uniformly integrable.

Each V_n is decreasing, thus $Z_n V_n^+(g/Z_n) \le Z_n V_n^+(f/Z_n)$, and $(Z_n V_n^+(f/Z_n))_{n\in\mathbb{N}}$ is $\mathbb{P}$-uniformly integrable by assumption.

On the other hand, since $V_n(x) \ge -x$ for all $n \in \mathbb{N}$ we get $Z_n V_n^-(g/Z_n) \le g$. The uniform integrability of $(Z_n V_n^-(g/Z_n))_{n\in\mathbb{N}}$ now follows from the fact that $g \in \mathbf{L}^1(\mathbb{P})$. □

2.3.2. *An upper semicontinuity-property of the sequence of the dual value functions.* We proceed here to show that for fixed $(y, r) \in \mathcal{L}$ we have

$$\limsup_{n\to\infty} v_n(y, r) \le v(y, r) \tag{2.10}$$

With $p := y^{-1}r$, pick some $\mathbb{Q} \in \mathcal{Q}'(p)$ such that $(Z_n V_n^+(y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}_n))_{n\in\mathbb{N}}$ is $\mathbb{P}$-uniformly integrable (observe that *this* is where we use our (UI) assumption). Then, $V^+(y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}) \in \mathbf{L}^1(\mathbb{P})$, and, according to Lemma 2.3, $v(y, r) = \inf_{g\in\mathcal{B}(y,r,\mathbb{Q})} \mathbb{E}[V(g)]$.

For any $g \in \mathcal{B}(y, r, \mathbb{Q}) \subseteq \mathcal{D}(y, r)$, Lemma 2.4 implies that $\lim_{n\to\infty} \mathbb{E}[Z_n V_n(g/Z_n)] = \mathbb{E}[V(g)]$; it follows that $\limsup_{n\to\infty} v_n(y_n, r_n) \le \limsup_{n\to\infty} \mathbb{E}[Z_n V_n(g/Z_n)] = \mathbb{E}[V(g)]$. Taking the infimum over all $g \in \mathcal{B}(y, r, \mathbb{Q})$ in the right-hand-side of the last inequality we arrive at (2.10).

2.4. **Limits of sequences of primal value functions and marginal utility-based prices.** In subsection 2.3 above, we established that $(v_n)_{n\in\mathbb{N}}$ converges pointwise to v on $\mathcal{L}$. Since all the functions involved are convex, the convergence is uniform on compact subsets of $\mathcal{L}$. Thanks to the strong stability properties of the family of convex functions on finite-dimensional spaces, this fact (and this fact only) yields convergence of the concave primal value functions, as well as the related sub-differentials and super-differentials to the corresponding limits. Indeed, by Theorem 7.17, p. 252 in [25], pointwise convergence on the interior of the effective domain of the limiting function is equivalent to the weaker notion of *epi-convergence*. In our case, primal value functions are all defined on $\mathcal{K}$ and the dual value functions on $\mathcal{L}$, both of which are, in fact, open thanks to the assumption (N-TRAD). For reader's convenience, we repeat the definition of epi-convergence

Definition 2.5 (Definition 7.1., p. 240, [25])**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of lower semicontinuous and proper convex functions defined on some Euclidean space $\mathbb{R}^d$. We say that f_n *epi-converges* to f — and write $f_n \overset{e}{\to} f$ if

(1) $\forall\, x \in \mathbb{R}^d,\ \forall\, x_n \to x$, $\liminf f_n(x_n) \ge f(x)$;
(2) $\forall\, x \in \mathbb{R}^d,\ \exists\, x_n \to x$, $\limsup f_n(x_n) \le f(x)$.

Epi-convergence seems to be tailor-made to interact well with conjugation. Denoting the convex conjugation by $(\cdot)^*$, Theorem 11.34., p. 500., in [25] states that

$$f_n \overset{e}{\to} f \ \Leftrightarrow\ f_n^* \overset{e}{\to} f^*,$$

as long as the functions $(f_n)_{n\in\mathbb{N}}$ are proper and lower-semicontinuous. An immediate consequence of the above fact is that

$$u(x,q) = \lim_n u_n(x,q), \text{ for all } (x,q) \in \mathcal{K}.$$

Moreover, the functions u_n as well as the limiting function u are know to be convex, so the stated pointwise convergence is, in fact, uniform on compacts. Therefore, the following, stronger, conclusion holds

$$u(x_n,q_n) = \lim_n u_n(x_n,q_n), \text{ for all } (x_n,q_n) \to (x,q) \in \mathcal{K}.$$

The list of pleasant properties of epi-convergence is not exhausted yet. By Theorem 12.35., p. 551 in [25], epi-convergence of convex functions implies the convergence of their sub-differentials, in the sense of graphical convergence, as defined below (the dimension $d \geq 1$ of the underlying space is general, but will be applied as $d = N+1$):

Definition 2.6 (Definition 5.32., p. 166, Proposition 5.33., p. 167)**.** Let $T, (T_n)_{n\in\mathbb{N}} : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a sequence of correspondences. We say that T_n *graphically converges* to T, and write $T_n \xrightarrow{g} T$, if for all $x \in \mathbb{R}^d$,

$$\bigcup_{\{x_n\to x\}} \limsup_n T_n(x_n) \subseteq T(x) \subseteq \bigcup_{\{x_n\to x\}} \liminf_n T_n(x_n),$$

where lim inf and lim sup should be interpreted in the usual set-theoretical sense, and the unions are taken over all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$, converging to x.

In order to combine the results mentioned above and illustrate the notion of graphical convergence in more familiar terms, we state and prove the following simple observation.

Proposition 2.7. *Suppose that $f_n \xrightarrow{e} f$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ converging towards some $x \in \mathbb{R}^d$. Then*

$$\limsup_n \partial f_n(x_n) \subseteq \partial f(x). \tag{2.11}$$

Further, let $\sup\{\|y\| \mid y \in \partial f_n(x_n),\ n \in \mathbb{N}\} < \infty$. Then, for each $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for $n \geq n(\varepsilon)$, $\partial f_n(x_n) \subseteq \partial f(x) + \varepsilon B$, with B denoting the unit ball of $\mathbb{R}^d$.

Proof. The first statement follows from the definition of graphical convergence, and its relationship to epi-convergence. For the second, suppose, to the contrary, that we can find an $\varepsilon > 0$ and an increasing sequence $n_k \in \mathbb{N}$ such that there exist points $x_k^* \in \partial f_{n_k}(x_{n_k})$ such that $d(x_k^*, \partial f(x)) > \varepsilon$. If $(x_k^*)_{k\in\mathbb{N}}$ has a convergent subsequence, then its limit x_0^* has to satisfy $d(x_0^*, \partial f(x)) \geq \varepsilon$ — a contradiction with (2.11). Therefore, there exists a subsequence of $(x_k^*)_{k\in\mathbb{N}}$, converging to $+\infty$ in norm. This, however, contradicts assumed uniform boundedness of subdifferentials. □

The sequences $(v_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ converge in a pointwise fashion, uniformly on compacts. It follows now directly from Definition 2.5 that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge in the epi sense towards u and v. As we have already mentioned above, epi convergence implies graphical convergence of the subdifferentials. To be able to use the additional conclusion of Proposition 2.7, we need to establish uniform boundedness of the superdifferentials of the functions $u(x_n, q_n; U_n, \mathbb{P}_n)$, when (x_n, q_n) live in a compact subset of $\mathcal{K}$. By Theorem 1.1, these are all of the form

$$\partial u(x_n, q_n; U_n, \mathbb{P}_n) = \{y_n\} \times \partial_q u(x_n, q_n; U_n, \mathbb{P}_n), \text{ where } y_n = \frac{\partial}{\partial x} u(x_n, q_n; U_n, \mathbb{P}_n).$$

It is an easy consequence of the second inclusion in the definition of the graphical convergence, and the differentiability in the x-direction of all functions u, $(u_n)_{n\in\mathbb{N}}$ that $y_n \to y = \frac{\partial}{\partial x} u(x, q; U, \mathbb{P})$. In particular, the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded away from zero, so, in order to use Proposition 2.7, it is enough to show that the sets $\mathcal{P}(x_n, q_n; U_n, \mathbb{P}_n)$ of utility-based prices are uniformly bounded. This fact follows immediately, once we recall that those are always contained in the sets of arbitrage-free prices, which are uniformly bounded by (S-REP). It remains to use Proposition 2.7 above and remember the characterization (1.2), to complete the proof of parts (1) and (3) of our main Theorem 1.6.

2.5. **Continuity of the optimal dual element and optimal wealth processes.** We conclude the proof of Theorem 1.6, tackling item (2) on convergence of the optimal terminal wealth and the optimal dual elements. Let $(x_n, q_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(x_n, q_n) =: (x, q) \in \mathcal{K}$ and $(y_n, r_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(y_n, r_n) =: (y, r) \in \mathcal{L}$ be, respectively, a $\mathcal{K}$-valued and an $\mathcal{L}$-valued sequence.

2.5.1. *Preparation.* Remember from Theorem 1.1 that the optimal dual and optimal primal elements are connected via

$$\hat{X}_T(x_n, q_n; U_n, \mathbb{P}_n) + \langle q_n, f\rangle = -V_n'(\hat{Y}_T(y_n, r_n; V_n, \mathbb{P}_n)), \text{ where } (y_n, r_n) \in \partial u(x_n, q_n).$$

If we show that $\lim_{n\to\infty} \hat{Y}_T(y_n, r_n; V_n, \mathbb{P}_n) = \hat{Y}_T(y, r; V, \mathbb{P})$ in $\mathbf{L}^0$ for all sequences $(y_n, r_n)_{n\in\mathbb{N}}$ that are $\mathcal{L}$-valued with $\lim_{n\to\infty}(y_n, r_n) =: (y, r) \in \mathcal{L}$, then the convergence of the random variables $\hat{X}_T(x_n, q_n; U_n, \mathbb{P}_n)$ to $\hat{X}_T(x, q; U, \mathbb{P})$ in $\mathbf{L}^0$ will follow as well. Indeed, fix a $\mathcal{K}$-valued sequence $(x_n, q_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(x_n, q_n) =: (x, q) \in \mathcal{K}$; from the upper hemicontinuity property proved in Subsection 2.4, we can choose for each $n \in \mathbb{N}$ some $(y_n, r_n) \in \partial u(x_n, q_n)$ in such a way as to have $\lim_{n\to\infty}(y_n, r_n) =: (y, r) \in \partial u(x, q) \subseteq \mathcal{L}$. The claim now follows easily; indeed, $\lim_{n\to\infty} \hat{Y}_T(y_n, r_n; V_n, \mathbb{P}_n) = \hat{Y}_T(y, r; V, \mathbb{P})$ in $\mathbf{L}^0$, $\hat{Y}_T(y, r; V, \mathbb{P}) > 0$, a.s., and $(V_n')_{n\in\mathbb{N}}$ converges uniformly to V' on compact subsets of $(0, \infty)$.

To ease notation we write $g_n := Z_n \hat{Y}_T(y_n, r_n; V_n, \mathbb{P}_n)$ and $g := \hat{Y}_T(y, r; V, \mathbb{P})$. Then $g_n \in \mathcal{D}(y_n, r_n)$ satisfies $v_n(y_n, r_n) = \mathbb{E}[Z_n V_n(g_n/Z_n)]$ for each $n \in \mathbb{N}$ and $g \in \mathcal{D}(y, r)$ satisfies $v(y, r) = \mathbb{E}[V(g)]$. The condition $\lim_{n\to\infty} \hat{Y}_T(y_n, r_n; V_n, \mathbb{P}_n) = \hat{Y}_T(y, r; V, \mathbb{P})$ in $\mathbf{L}^0$ that we need to prove translates to $\lim_{n\to\infty}(g_n/Z_n) = g$ in $\mathbf{L}^0$. Assume that an arbitrary subsequence (whose indices are not relabeled) has already been extracted from $(g_n/Z_n)_{n\in\mathbb{N}}$. It suffices to show that $\lim_{k\to\infty}(g_{n_k}/Z_{n_k}) = g$ in $\mathbf{L}^0$ along some further *subsequence* $(g_{n_k}/Z_{n_k})_{k\in\mathbb{N}}$.

We now make a further step that — although seemingly confusing — will prove useful in the sequel of our proof. With $p := y^{-1}r$, we pick some $\mathbb{Q} \in \mathcal{Q}(p)$ such that $f := y\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ satisfies $V^+(f) \in \mathbf{L}^1(\mathbb{P})$. For each $n \in \mathbb{N}$ let $f_n := n^{-1}f + (1-n^{-1})g$. Note that $f_n \in \mathcal{B}(y, r, \mathbb{Q})$ for all $n \in \mathbb{N}$, with $\mathcal{B}(y, r, \mathbb{Q})$ as in (2.9), and that $\lim_{n\to\infty} f_n = g$ in $\mathbf{L}^0$.

For any $m \in \mathbb{N}$ define

$$C_m := \{(a, b) \in \mathbb{R}^2 \mid 1/m \le a \le m,\ 1/m \le b \le m,\ \text{and } |a - b| > 1/m\}. \tag{2.12}$$

Combining the discussion above with the facts that $\lim_{n\to\infty} Z_n = 1$ in $\mathbf{L}^0$ and that both sequences $(g_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are bounded in $\mathbf{L}^0$, we conclude that in order to prove that $\lim_{n\to\infty}(g_n/Z_n) = g$ in $\mathbf{L}^0$, we need to establish the following claim:

Claim 2.8. *There exist a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ so that the subsequences $(g_{n_m})_{m\in\mathbb{N}}$ and $(f_{n_m})_{m\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ respectively, satisfy*

$$\lim_{m\to\infty} \mathbb{P}_{n_m}\left[(g_{n_m}/Z_{n_m}, f_{n_m}/Z_{n_m}) \in C_m\right] = 0. \tag{2.13}$$

The above clarifies the reason why the sets C_m, $m \in \mathbb{N}$ of (2.12) were introduced; in fact, this trick is a more elaborate version of the method used in the proof of Lemma A.1 of [6].

Remark 2.9. For a sequence $(A_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$-measurable sets, $\lim_{n\to\infty} \mathbb{P}_n[A_n] = 0$ is equivalent to $\lim_{n\to\infty} Z_n\mathbb{I}_{A_n} = 0$ in $\mathbf{L}^0$ (combining the $\mathbf{L}^1(\mathbb{P})$-convergence of the last sequence with $\mathbb{P}$-uniform integrability of $(\mathbb{P}_n)_{n\in\mathbb{N}}$) which, in view of the fact $\lim_{n\to\infty} Z_n = 1$ in $\mathbf{L}^0$, is equivalent to $\lim_{n\to\infty} \mathbb{I}_{A_n} = 0$ in $\mathbf{L}^0$, or in other words that $\lim_{n\to\infty} \mathbb{P}[A_n] = 0$. This justifies the use of "$\mathbb{P}_{n_m}$" instead of "$\mathbb{P}$" in (2.13).

2.5.2. *Proof of Claim 2.8.* For any $m \in \mathbb{N}$, the strict convexity of V implies the existence of some $\beta_m > 0$ such that for all $(a, b) \in (0, \infty)^2$ we have

$$V\Big(\frac{a+b}{2}\Big) \le \frac{V(a) + V(b)}{2} - \beta_m \mathbb{I}_{C_m}(a, b), \text{ for the set } C_m \text{ of (2.12).}$$

Uniform convergence of $(V_n)_{n\in\mathbb{N}}$ to V on compact subsets of $(0, \infty)$ implies that (with a possible lower, but still strictly positive, choice of β_m) we still have

$$V_n\Big(\frac{a+b}{2}\Big) \le \frac{V_n(a) + V_n(b)}{2} - \beta_m \mathbb{I}_{C_m}(a, b),$$

for all $n \in \mathbb{N}$ and $(a, b) \in (0, \infty)^2$. Setting $a = g_n/Z_n$, $b = f_k/Z_n$, multiplying both sides of the previous inequality with Z_n, and taking expectation with respect to $\mathbb{P}$, one gets

$$\begin{aligned}
\beta_m \mathbb{P}_n\Big[\Big(\frac{g_n}{Z_n}, \frac{f_k}{Z_n}\Big) \in C_m\Big] &\le \frac{1}{2}\mathbb{E}\Big[Z_n V_n\Big(\frac{g_n}{Z_n}\Big)\Big] + \frac{1}{2}\mathbb{E}\Big[Z_n V_n\Big(\frac{f_k}{Z_n}\Big)\Big] - \mathbb{E}\Big[Z_n V_n\Big(\frac{g_n + f_k}{2Z_n}\Big)\Big] \\
&\le \frac{1}{2} v_n(y_n, r_n) + \frac{1}{2}\mathbb{E}\Big[Z_n V_n\Big(\frac{f_k}{Z_n}\Big)\Big] - v_n\Big(\frac{y_n + y}{2}, \frac{r_n + r}{2}\Big),
\end{aligned}$$

where for the third term of the last inequality we have used that fact that

$$\frac{g_n + f_k}{2Z_n} \in \mathcal{D}\Big(\frac{y_n + y}{2}, \frac{r_n + r}{2}\Big).$$

Invoking Lemma 2.4, we know that $\lim_{n\to\infty} \mathbb{E}[Z_n V_n(f_k/Z_n)] = \mathbb{E}[V(f_k)]$ for all fixed $k \in \mathbb{N}$. Furthermore, the proof of Lemma 2.3 shows that $\lim_{k\to\infty} \mathbb{E}[V(f_k)] = \mathbb{E}[V(g)] = v(y,r)$. Using also the uniform convergence (on compact subsets of $\mathcal{L}$) of $(v_n)_{n\in\mathbb{N}}$ to v, we see that we can choose k_m and n_m large enough so that $\mathbb{P}_{n_m}[Z_{n_m}^{-1}(g_{n_m}, f_{k_m}) \in C_m] \leq 1/m$. It is a matter of subsequence manipulation to show that one can, in fact, choose a universal strictly increasing sequence $n_m = k_m$, $m \in \mathbb{N}$ with all the desired properties. This proves (2.13), and concludes the proof of our main Theorem 1.6.

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Constantinos Kardaras, Mathematics and Statistics Department, Boston University, 111 Cummington Street, Boston, MA 02215, USA.

E-mail address: kardaras@bu.edu

Gordan Žitković, Department of Mathematics, University of Texas at Austin, 1 University Station, C1200, Austin, TX 78712, USA

E-mail address: gordanz@math.utexas.edu